On the Drawing by Mathematica

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Abstract

In this paper we consider a rotation which makes a straight line in the space a shaft and express it by base change of a linear space. This idea will be used to make algorithm, and we will show a couple of programs of Mathematica as examples.

1 Preface

In [1] we showed the animated program of Mathematica which assembled a proper polyhedron from the development figure. But the rotation which made a straight line in the space a shaft showed a problem there, and it was a part. Though the rotation of the problem was realized by combining several rotations there, the way for example needing considerable time for the preparation of the proper dodecahedron and dividing it into several steps and making it was taken. This reason can be thought rotation about the straight line to be because a complex expression was taken. As pointed in [1], it could be thought about the method which a base change was used for a rotation about the straight line. It couldn't be put into action for the various circumstances though the author wanted to do if he had time. Since the animated programs were stopped running as the result the version of Mathematica was upgraded in 6 and 7, the author had to rewrite programs for the new versions. On this occasion, the author decided to use a base change to express the rotation about a straight line with the renewal of the animated programs. When this was done, it found that algorithm was so simple, too, and it could be carried out far more efficiently than [1]. It can deal with contents here in the level of linear algebra of the culture of the university. So we think that the contents here can be used as a subject.

2 Base change and rotation

Let $\{\boldsymbol{e_1},\ \boldsymbol{e_2},\ \boldsymbol{e_3}\}$ be the standard basis of the space. Let $\{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$ be another orthonormal basis. Then the matrix $M = (\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w})$ is the one which expresses the base change $\{\boldsymbol{e_1},\ \boldsymbol{e_2},\ \boldsymbol{e_3},\ \} \to \{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$. Let $X,\ Y,\ Z$ be the axes corresponding to $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}$ respectively. The rotation of an angle θ with respect to the X-axis is $Rot = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ if we denote it by an matrix with respect to the basis $\{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$. Suppose a point $\mathrm{P}(x,\ y,\ z)$ is mapped to $\mathrm{Q}(x',\ y',\ z')$ by the rotation with respect to the X-axis of the angle θ. Note that $\overrightarrow{\mathrm{OP}} = x\boldsymbol{e_1} + y\boldsymbol{e_2} + z\boldsymbol{e_3}$. Let $\overrightarrow{\mathrm{OP}} = X\boldsymbol{u} + Y\boldsymbol{v} + Z\boldsymbol{w}$ if we denote it with respect to the basis $\{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$. Thus we note that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$. Let $\overrightarrow{\mathrm{OQ}} = X'\boldsymbol{u} + Y'\boldsymbol{v} + Z'\boldsymbol{w}$. Then $\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = Rot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$. Since $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$, it holds that $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M\,Rot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M\,Rot\,M^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since M is an orthogonal matrix, $M^{-1} = {}^tM$ holds. Hence the matrix which expresses the rotation with respect to the X-axis of the angle θ is given by $M\,Rot\,{}^tM$ with respect to the standard basis.

3 On the way to decide the orthonormal basis when a rotation shaft is specified

In this section we use the result of §2 in order to find the matrix which expresses the rotation of angle θ that makes a straight line ℓ a shaft.

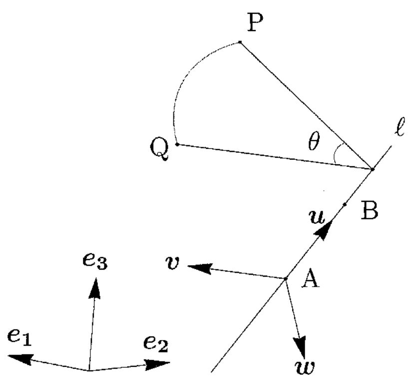
Let $a\boldsymbol{e}_1+b\boldsymbol{e}_2+c\boldsymbol{e}_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the direction vector of ℓ and put $\boldsymbol{u} = \dfrac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Let π be the flat plane which contains the origin and is perpendicular to ℓ. Then the equation of π is $ax+by+cz=0$. It should be noted that $(b,\ -a,\ 0)$ is a point of π if $a \neq 0$ or $b \neq 0$, and in this case let us put $\boldsymbol{v} = \dfrac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} = \begin{pmatrix} ac \\ bc \\ -(a^2+b^2) \end{pmatrix}$, we put $\boldsymbol{w} = \dfrac{1}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \begin{pmatrix} ac \\ bc \\ -(a^2+b^2) \end{pmatrix}$. If $a=b=0$ then $(1,\ 1,\ 0)$ is a point of π. Hence we put $\boldsymbol{v} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. In this case we put $\boldsymbol{w} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. In either case we obtained an orthonormal basis $\{\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\}$.

The line ℓ does not need to contain the origin. So we fix a point (say) $\mathrm{A}(\alpha,\ \beta,\ \gamma)$ on ℓ. When a rotation of angle θ which makes the line ℓ a shaft is done, suppose a point $\mathrm{P}(x,\ y,\ z)$ is mapped to a point $\mathrm{Q}(x',\ y',\ z')$. Let φ_1 be the parallel transformation such as $\varphi_1(\boldsymbol{t}) = \boldsymbol{t} - \overrightarrow{\mathrm{OA}}$. Then ℓ is mapped to the line which contains the origin by φ_1. Let φ_2 be the rotation of angle θ which makes the X-axis a shaft. Note that $\varphi_1^{-1}(\boldsymbol{t}) = \boldsymbol{t} + \overrightarrow{\mathrm{OA}}$. Then the combination $\varphi_1^{-1} \circ \varphi_2 \circ \varphi_1$ is the same as the rotation of angle θ which makes the line ℓ a shaft. Let matrices M and Rot be the same as in §2. Then φ_2 corresponds to $M\,Rot\,{}^tM$ by §2. Therefore $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M\,Rot\,{}^tM \begin{pmatrix} x-\alpha \\ y-\beta \\ z-\gamma \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ holds.

4 Examples

Example1 The program of the rotation of angle $\frac{\pi}{3}$ which makes the line containing two points $\mathrm{A}(-2,\ 3,\ 2)$, $\mathrm{B}(-1,\ 5,\ 3)$ a shaft and the result of practice.

```
a = {-2, 3, 2}; b = {-1, 5, 3}; q = {3.5, 2, -3}; theta = Pi/3;
d1 = Sqrt[(b[[1]] - a[[1]])^2 + (b[[2]] - a[[2]])^2 + (b[[3]] - a[[3]])^2];
u = {b[[1]] - a[[1]], b[[2]] - a[[2]], b[[3]] - a[[3]]}/d1;
If[b[[1]] - a[[1]]==0&&b[[2]] - a[[2]]==0, v = {1, 1, 0}/Sqrt[2];
w = {1, -1, 0}/Sqrt[2],
d2 = Sqrt[(b[[1]] - a[[1]])^2 + (b[[2]] - a[[2]])^2];
v = {b[[2]] - a[[2]], -(b[[1]] - a[[1]]), 0}/d2;
d3 = Sqrt[((b[[1]] - a[[1]])^2 + (b[[2]] - a[[2]])^2)
((b[[3]] - a[[3]])^2 + (b[[2]] - a[[2]])^2 + (b[[1]] - a[[1]])^2)];
w = {(b[[3]] - a[[3]])(b[[1]] - a[[1]]),
        (b[[3]] - a[[3]])(b[[2]] - a[[2]]),
        -((b[[1]] - a[[1]])^2 + (b[[2]] - a[[2]])^2)}/d3 ];
m1 = Transpose[{u, v, w}];
l1 = Graphics3D[Line[{a - 1.5(b - a), a + 1.7(b - a)}]];
pa = Graphics3D[Point[a]];
pb = Graphics3D[Point[b]];
p1 = Graphics3D[Point[m1.q + a]];
p2 = Graphics3D[Point[m1.{q[[1]], 0, 0} + a]];
m2 = {{1, 0, 0}, {0, Cos[theta], -Sin[theta]},
{0, Sin[theta], Cos[theta]}};
m3 = {{1, 0, 0}, {0, Cos[t], -Sin[t]}, {0, Sin[t], Cos[t]}};
p3 = Graphics3D[Point[m1.m2.q + a]];
l2 = Graphics3D[Line[{m1.{q[[1]], 0, 0} + a, m1.q + a}]];
l3 = Graphics3D[Line[{m1.{q[[1]], 0, 0} + a, m1.m2.q + a}]];
c = ParametricPlot3D[m1.m3.q + a, {t, 0, theta}];
c2 = ParametricPlot3D[m1.m3.{3.5, 2/5, -3/5} + a, {t, 0, theta}];
ar1 = Graphics3D[Arrow[{{0, 0, 0}, {2, 0, 0}}]];
ar2 = Graphics3D[Arrow[{{0, 0, 0}, {0, 2, 0}}]];
ar3 = Graphics3D[Arrow[{{0, 0, 0}, {0, 0, 2}}]];
br1 = Graphics3D[Arrow[{a, 2u + a}]];
br2 = Graphics3D[Arrow[{a, 2v + a}]];
br3 = Graphics3D[Arrow[{a, 2w + a}]];
Show[pa, pb, l1, l2, l3, p1, p2, p3, ar1, ar2, ar3, br1, br2, br3,
        c, c2, Boxed → False]
```



(In the figure, symbols such as vectors, points etc. are added to make it easy to see.)

Example2 The movement of the fixed star and the planet. A fixed star is in the oval focus, the planet which makes the oval a revolution orbit, the animated program which revolves with the rotation and which slants the planet's axis to the direction of $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

```
rot1 = {{Cos[kPi/100], −Sin[kPi/100], 0}, {Sin[kPi/100], Cos[kPi/100], 0},
{0, 0, 1}};
rot2 = {{Cos[kPi/10], Sin[kPi/10], 0}, {−Sin[kPi/10], Cos[kPi/10], 0}, {0, 0, 1}};
{x1, y1, z1} = rot1.{Sin[theta1]Cos[phi1], Sin[theta1]Sin[phi1], Cos[theta1]}−
{2, 0, 0};
ball1:=ParametricPlot3D[{x1, y1, z1}, {theta1, 0, Pi}, {phi1, 0, 2Pi}];
{a, b, c} = {1, 0, 4};
m = Transpose[{1/Sqrt[a^2 + b^2]{b, −a, 0},
1/Sqrt[(a^2 + b^2)(a^2 + b^2 + c^2)]{ac, bc, −(a^2 + b^2)},
1/Sqrt[a^2 + b^2 + c^2]{a, b, c}}];
r = 6/(Cos[kPi/100] + 2);
{x2, y2, z2} =
m.rot2.{2/3Sin[theta2]Cos[phi2], 2/3Sin[theta2]Sin[phi2], 4/7Cos[theta2]}+
{rCos[kPi/100] + 2, rSin[kPi/100], 0};
ball2:=ParametricPlot3D[{x2, y2, z2}, {theta2, 0, Pi}, {phi2, 0, 2Pi}];
ListAnimate[
Table[Show[ball1, ball2, Axes → False, Boxed → False, ViewPoint → {0, −5, 1},
PlotRange → {{−4.5, 4.5}, {−4, 4}, {−1.5, 1.5}}], {k, 0, 199}]]
```

Remark If we delete the last ListAnimate[ ], the program will go and end by drawing 200 figures. We can save these figures as GIF images. We can also use Giam.exe (made by Mr. Furumizo) to make AnimeGif file.

References

[1] K. Ohtake, On animations to construct regular polyhedra by Mathematica from their developments (in Japanese), Science Reports of the Faculty of Education Gunma University, Vol. 54(2006), 1-12.

[2] K. Ohtake and H. Fukushima, "A First Course in Linear Algebra" (in Japanese), Makino Shoten, 2003.